

SMITH NORMAL FORM OF MATRICES ASSOCIATED WITH DIFFERENTIAL POSETS

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ABSTRACT. We prove a conjecture of Miller and Reiner on the Smith Normal Form of the operator DU_n associated with a differential poset under certain conditions. We do so by proving that the conjecture is equivalent to the existence of a certain decomposition of the $\mathbb{Z}[x]$ -module (\mathbb{Z}^{P_n}, DU_n) , similar to the invariant factor decomposition of $\mathbb{Q}[x]$ -modules. We then use our result to verify the conjecture in certain special cases.

1. INTRODUCTION

Let r be a positive integer. We say that a poset P is r -differential, if it satisfies the following three conditions:

- (D1) P is graded, locally finite, has all rank sizes finite and has a unique minimum element.
- (D2) If two distinct elements of P have exactly k elements that they both cover, then there will be exactly k elements that cover them both.
- (D3) If an element of P covers k elements, then it will be covered by $k + r$ elements.

Associated to every r -differential poset are two families of maps, known as *up* and *down* maps. Let P_n be the n -th rank of P , which we take to be the empty set if $n < 0$, and $p_n = |P_n|$. For any commutative ring R with identity and characteristic 0, let $RP_n \cong R^{p_n}$ be the free module over R with basis P_n . We define

$$\begin{aligned} U_n : RP_n &\rightarrow RP_{n+1} \\ D_n : RP_n &\rightarrow RP_{n-1} \end{aligned}$$

for all $n \geq 0$ by saying that U_n sends an element in P_n to sum, with coefficients 1, of all elements in P_{n+1} that cover that element, and D_n sends an element in P_n to the sum, with coefficients 1, of all the elements in P_{n-1} that are covered by it. We then define

$$\begin{aligned} UD_n &:= U_{n-1}D_n \\ DU_n &:= D_{n+1}U_n \end{aligned}$$

The two conditions (D2) and (D3) can then be recast as

$$DU_n - UD_n = rI$$

The most well-known examples of 1-differential posets are \mathbf{Y} , the Young's Lattice and \mathbf{YF} , the Young-Fibonacci Lattice. Their r -fold cartesian products, denoted by \mathbf{Y}^r and $Z(r)$ respectively, are examples of r -differential posets.

Differential posets were first defined by Stanley in [4], with the up and down maps defined over fields. Later, Miller and Reiner defined them over arbitrary rings in [3], as

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we have done here, and conjectured a remarkable property of the DU_n operators over the ring of integers which we now describe.

Let $A = (a_{ij})$ be a $m \times n$ matrix over R . We say that A has *Smith Normal Form* (SNF) over R if there exist invertible matrices $P \in R^{m \times m}$, $Q \in R^{n \times n}$ such that $B = PAQ$ is a diagonal matrix, in the sense that $b_{ij} = 0$ if $i \neq j$, and $s_i := b_{ii}$ for $1 \leq i \leq k = \min\{m, n\}$ satisfy $s_1 | s_2 | \dots | s_k$. It is known that if R is a PID, any matrix A always has a SNF which is unique, in the sense that the diagonal entries s_i are unique up to units of R . If R is not a PID, then a SNF may not necessarily exist. However, it is unique if it does exist.

Assume now that $R = \mathbb{Z}$. Let $[DU_n]$ be the matrix of DU_n with respect to the standard basis of $\mathbb{Z}P_n$ and I_{p_n} be the $p_n \times p_n$ identity matrix.

Conjecture 1.1. [3, Miller-Reiner] *For any differential poset P , and any $n \geq 0$, the matrix $[DU_n] + xI_{p_n}$ has SNF over $\mathbb{Z}[x]$.*

Miller and Reiner verified this conjecture for the r -differential posets $Z(r)$ in [3]. Recently, the problem was investigated by Cai and Stanley in [1] for the case \mathbf{Y}^r and the case $r = 1$ was settled in the affirmative. As noted at the end of their paper, the case $r > 1$ was later handled by Zipei Nie.

In this paper, we prove this conjecture for any r -differential poset that satisfies certain conditions, as stated in Theorem 4.5. We do so by looking at the $\mathbb{Z}[x]$ -module structure of $\mathbb{Z}P_n$, where the action of x is induced by the operator DU_n . The conditions assumed are closely related to two additional conjectures (2.3 and 2.4) made by Miller and Reiner in [3]. We then verify this conjecture for a fairly general class of cartesian products of differential posets, and use this result to deduce the previously studied cases of $Z(r)$ and \mathbf{Y}^r as straightforward implications.

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2. $\mathbb{Q}[x]$ -MODULES AND DU OPERATORS

We start with a theorem of Stanley.

Theorem 2.1. [4, §4] *Let P be a r -differential poset and R a field of characteristic 0. Then*

$$\begin{aligned} \text{Ch}(DU_n) &= \prod_{i=0}^n (x - r(i+1))^{\Delta_{p_n-i}} \\ \text{Ch}(UD_n) &= \prod_{i=0}^n (x - ri)^{\Delta_{p_n-i}} \end{aligned}$$

where $\text{Ch}(A) = \text{Ch}(A, x)$ denotes the characteristic polynomial of the operator A , and $\Delta_{p_n} := p_n - p_{n-1}$. Furthermore, the operators DU_n and UD_n are diagonalizable.

We make some immediate conclusions. First, the rank sizes increase weakly, as Δp_n must be non-negative. Second, the spectra of DU_n tells us that it is invertible. Thus, U_n is injective and D_{n+1} is surjective for all $n \geq 0$.

Assume now that $R = \mathbb{Q}$. For each n , we view \mathbb{Q}^{p_n} as a $\mathbb{Q}[x]$ -module, with x -action induced by DU_n . Then, as DU_n is diagonalizable, the $\mathbb{Q}[x]$ -module structure of \mathbb{Q}^{p_n} as described by the invariant factor decomposition (IFD) theorem for PIDs (in our case, $\mathbb{Q}[x]$) is easy to describe: there exist $m = \max \{\Delta p_j\}_{j=0}^n$ cyclic $\mathbb{Q}[x]$ -submodules of \mathbb{Q}^{p_n} , say V_1, V_2, \dots, V_m with annihilators

$$a_i(x) = \prod_{\substack{1 \leq j \leq n+1 \\ \Delta p_{n+1-j} \geq m-i+1}} (x - r_j) \quad 1 \leq i \leq m$$

such that $\mathbb{Q}^{p_n} = V_1 \oplus V_2 \oplus \dots \oplus V_m$. Note that $a_1(x)|a_2(x)|\dots|a_m(x)$. Since the non-constant polynomials in the SNF of $xI_{p_n} - [DU_n]_\beta$ for any basis β are the invariant factors of \mathbb{Q}^{p_n} , the SNF is

$$\text{diag}(1, \dots, 1, a_1(x), a_2(x), \dots, a_m(x)).$$

See [2, §12.2] for details.

We now analyze what happens when we take $R = \mathbb{Z}$. First, since the matrix of DU_n and UD_n in the standard basis of \mathbb{Z}^{p_n} and \mathbb{Q}^{p_n} are the same, the charactersitic polynomials remain the same. Injectivity of U_n and DU_n still holds, since injectivity of a matrix over \mathbb{Z} trivially follows from injectivity over \mathbb{Q} .

However, DU_n and UD_n are not necessarily diagonalizable over \mathbb{Z} in the sense that there is no basis of \mathbb{Z}^{p_n} such that the matrix of DU_n (or UD_n) with respect to it are diagonal matrices.

An explicit example is given by DU_2 for $P = \mathbf{Y}$. In the standard basis the matrix, is $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Diagonalizability of A is equivalent to requiring that $\mathbb{Z}^2 = \ker(DU_2 - I_2) \oplus \ker(DU_2 - 3I_2)$, or that

$$\mathbb{Z}^2 = \langle (1, -1) \rangle_{\mathbb{Z}} \oplus \langle (1, 1) \rangle_{\mathbb{Z}}$$

which is absurd, as $(1, 0)$ does not belong to the right hand side. Invertibility of DU_n is also no longer guranteed as the determinant of its matrix may not be ± 1 . For instance, $\det(A) = 3$, so DU_2 for \mathbf{Y} is not invertible. Thus, DU_n is not necessarily surjective. Similarly, one cannot say if down maps are still surjective

Finally, since $\mathbb{Z}[x]$ is not a PID, one in general does not expect a similar decomposition to exist for \mathbb{Z}^{p_n} , when viewed as a $\mathbb{Z}[x]$ -module with x -action induced by DU_n . However, we show that the existence of such a decomposition is equivalent to the conjecture of Miller and Reiner. We develop this concept in the next section.

3. $\mathbb{Z}[x]$ -MODULES AND RATIONAL CANONICAL FORM

Recall that a $\mathbb{Z}[x]$ -module M is just a pair (M, φ) where M is a \mathbb{Z} -module and $\varphi \in \text{End}_{\mathbb{Z}}(M, M)$. If M is cyclic, then $\mathbb{Z}[x]/\text{Ann}(M) \cong M$. If, in addition, the annihilator is a principal ideal, and can be generated by some monic, non-constant polynomial $a(x)$, then M is a free \mathbb{Z} -module of rank $d := \deg(a(x))$, with a basis given by $w, xw, \dots, x^{d-1}w$, where w is any $\mathbb{Z}[x]$ -generator for M .

Assume now that M is also a free \mathbb{Z} -module of finite rank. Then, we say that M has *rational canonical form over \mathbb{Z}* (RCF over \mathbb{Z}), if there exist cyclic $\mathbb{Z}[x]$ -modules V_1, V_2, \dots, V_k with annihilators $(a_1(x)), (a_2(x)), \dots, (a_k(x))$ such that $a_i(x)$ are monic,

non-constant, $a_1(x)|a_2(x)|\cdots|a_k(x)$ and $M = V_1 \oplus V_2 \oplus \cdots \oplus V_k$. Tensoring both sides with \mathbb{Q} over \mathbb{Z} , we see that $M \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}[x]$ -module, and is equal to the direct sum $(V_1 \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus (V_2 \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus \cdots \oplus (V_k \otimes_{\mathbb{Z}} \mathbb{Q})$, with each $V_i \otimes_{\mathbb{Z}} \mathbb{Q}$ cyclic and with annihilator $a_i(x)$. The IFD Theorem for $\mathbb{Q}[x]$ modules then gives us that the polynomials $a_i(x)$ are also unique, should they exist.

For a monic polynomial $a(x)$, we denote its companion matrix by $\mathcal{C}_{a(x)}$. Let $d_i := \deg a_i(x)$, v_i be a $\mathbb{Z}[x]$ -generator for V_i and

$$\alpha := \left\{ v_1, \dots, x^{d_1-1}v_1, v_2, \dots, x^{d_2-1}v_2, \dots, v_k, \dots, x^{d_k-1}v_k \right\}.$$

Then, α is a \mathbb{Z} -basis for M , and with respect to this basis, the matrix of action of x is a direct sum of k block matrices, the i -th block being the companion matrix of $a_i(x)$. In other words, the matrix of the action of x in the basis α is

$$\begin{pmatrix} \mathcal{C}_{a_1(x)} & & & \\ & \mathcal{C}_{a_2(x)} & & \\ & & \ddots & \\ & & & \mathcal{C}_{a_k(x)} \end{pmatrix}$$

We say that a $m \times m$ integer matrix has RCF over \mathbb{Z} if it is $\text{GL}(m, \mathbb{Z})$ conjugate to its RCF over \mathbb{Q} , and the RCF is itself an integer matrix. Similarly, an endomorphism on a free \mathbb{Z} -module of finite rank has RCF over \mathbb{Z} if the matrix of that endomorphism with respect to some basis has RCF over \mathbb{Z} . It is easily seen that M has RCF over \mathbb{Z} if and only if φ has RCF over \mathbb{Z} .

Theorem 3.1. *Let M be a free \mathbb{Z} -module of finite rank m and let β be a basis of M . Let $\varphi \in \text{End}_{\mathbb{Z}}(M, M)$, and consider $M = (M, \varphi)$ as a $\mathbb{Z}[x]$ -module. Then, M has RCF over \mathbb{Z} if and only if the matrix $A(x) = xI_m - [\varphi]_{\beta}$ has SNF over $\mathbb{Z}[x]$.*

Proof. Suppose that M has RCF over \mathbb{Z} . This is equivalent to saying that there is a basis α such that $[\varphi]_{\alpha}$ is in RCF. We can replace the basis β with α since, if $P(x)A(x)Q(x)$ puts $A(x)$ in SNF, choosing $S = [\text{id}_M]_{\alpha}^{\beta}$, one finds $P(x)S(xI_m - [\varphi]_{\alpha})S^{-1}Q(x)$ gives the same SNF, and vice versa. So, we can assume that the matrix $A(x)$ is

$$\begin{pmatrix} xI_{d_1} - \mathcal{C}_{a_1(x)} & & & \\ & xI_{d_2} - \mathcal{C}_{a_2(x)} & & \\ & & \ddots & \\ & & & xI_{d_k} - \mathcal{C}_{a_k(x)} \end{pmatrix}$$

where $d_i = \deg a_i(x)$. The SNF of $\mathcal{C}_{a_i(x)}$ is $(1, \dots, 1, a_i(x))$ and can be obtained by a sequence of simple row and column operations. Thus, one can convert the above matrix into $\text{diag}(1, \dots, a_1(x), 1, \dots, a_2(x), \dots, 1, \dots, a_k(x))$, and by applying a few row and column operations of switching, one obtains the desired SNF.

For the other direction, we essentially mimick the sequence of exercises 22-25 of [2, §12.2]. Let $\mathcal{M} = \mathbb{Z}[x]^m$, and let e_i be the standard $\mathbb{Z}[x]$ -basis for \mathcal{M} . Define a map $\chi : \mathcal{M} \rightarrow M$, which sends e_i to $b_i \in \beta$. Then, χ is surjective, and thus, $\mathcal{M}/\ker \chi \cong M$. Suppose that $[\varphi]_{\beta} = (a_{i,j})$. We exhibit an explicit set of generators for $\ker \chi$.

Let

$$v_j = -a_{1,j}e_1 - \dots - a_{j-1,j}e_{j-1} + (x - a_{j,j})e_j - a_{j+1,j}e_{j+1} - \dots - a_{m,j}e_m,$$

for $j \in \{1, 2, \dots, m\}$. It is easy to check that $v_j \in \ker \chi$. We claim v_j are $\mathbb{Z}[x]$ -generators for $\ker \chi$. To this end, notice that

$$xe_j = v_j + f_j,$$

where $f_j = -a_{1,j}e_1 - \dots - a_{m,j}e_m \in \mathbb{Z}e_1 + \dots + \mathbb{Z}e_m$. By, repeatedly applying these relations, we can show that

$$\mathcal{M} = \mathbb{Z}[x]^m = \mathbb{Z}[x]e_1 + \dots + \mathbb{Z}[x]e_m = \mathbb{Z}[x]v_1 + \dots + \mathbb{Z}[x]v_m + \mathbb{Z}e_1 + \dots + \mathbb{Z}e_m.$$

Thus, every element of \mathcal{M} can be written as a sum of an element of $\mathbb{Z}[x]$ -submodule V generated by v_1, v_2, \dots, v_m , and an element of \mathbb{Z} -module W generated by e_1, e_2, \dots, e_m in $\mathbb{Z}[x]^m$. Suppose now that $k \in \ker \chi$. Then, $k = v + w$ for some $v \in V$, $w \in W$. Then, $0 = \chi(k) = \chi(v + w) = \chi(v) + \chi(w) = \chi(w)$. However, $\chi(w)$ is 0 if and only if $w = 0$, as the elements b_1, b_2, \dots, b_m form a basis of M . Thus, the elements v_j generate $\ker \chi$ as a $\mathbb{Z}[x]$ -submodule.

Consider now the matrix $A(x)$. It's i -th column is the coordinate vector of v_i with respect to the standard basis e_i of \mathcal{M} . Notice that right multiplication of $A(x)$ by elements of $\text{GL}(m, \mathbb{Z}[x])$ can be interpreted as changing the set of generators of $\ker \chi$ and left multiplication as changing the basis of \mathcal{M} . So, saying that $A(x)$ has SNF over $\mathbb{Z}[x]$ is equivalent to saying that there is basis of \mathcal{M} , say $\hat{e}_1, \hat{e}_2, \dots, \hat{e}_m$ and integer polynomials $a_i(x)$ satisfying $a_1(x)|a_2(x)|\dots|a_m(x)$ such that $a_i(x)\hat{e}_i$ form a set of generators of $\ker \chi$. This implies that $\ker \chi$ is a free $\mathbb{Z}[x]$ -module and

$$\begin{aligned} M &\cong \mathcal{M} / \ker \chi \\ &\cong \mathbb{Z}[x]\hat{e}_1 \oplus \dots \oplus \mathbb{Z}[x]\hat{e}_m / (\mathbb{Z}[x]a_1(x)\hat{e}_1 \oplus \dots \oplus \mathbb{Z}[x]a_m(x)\hat{e}_m) \\ &\cong \mathbb{Z}[x]/(a_1(x)) \oplus \mathbb{Z}[x]/(a_2(x)) \oplus \dots \oplus \mathbb{Z}[x]/(a_m(x)). \end{aligned}$$

If some $a_i(x)$ is 1, we can remove it, since, then, $\mathbb{Z}[x]/(a_i(x)) = 0$. All of the $a_i(x)$ are monic, hence also, non-constant, since the product of $a_i(x)$ is the determinant of $A(x)$, which is monic. \square

By replacing x with $-x$ in the theorem above, we obtain

Corollary 3.2. *For a r -differential poset, the Conjecture 1.1 is true for some n if and only if DU_n has RCF over \mathbb{Z} .*

4. THE MAIN THEOREM

We need some preliminary work.

Let A be a $m \times n$ integer matrix and $k = \min\{m, n\}$. By $\Delta_S(A)$, we mean the k -tuple of the diagonal entries of SNF of A , which we can define uniquely by taking them to be all non-negative. If φ is a homomorphism of free \mathbb{Z} -modules of finite rank, then by $\Delta_S(\varphi)$, we mean the diagonal entries of the SNF of a matrix of φ .

Let M be a free \mathbb{Z} -module of finite rank. If N is a submodule of M , then N splits off as a direct summand of M if and only if $\Delta_S(N \hookrightarrow M)$ consists of $\text{rank}(N)$ 1s. In particular, an element of M can be extended to a basis M if and only if the gcd of its coefficients in some basis of M is 1.

For $b \in M$, and positive integer p , we say that $p|b$ if there is some $v \in M$ such that $pv = b$. With this terminology and the discussion above, we have

Lemma 4.1. *Let M be a free \mathbb{Z} -module of finite rank. Suppose that $b_1, b_2, \dots, b_k \in M$ are vectors that can be extended to a basis of M . Let $b = c_1b_1 + c_2b_2 + \dots + c_kb_k$ with $\gcd(c_1, c_2, \dots, c_k) = 1$. Then, there is no non-unit integer p such that $p|b$.*

Suppose now that $\text{rank}(N) = n$, $\text{rank}(M) = m$ and N splits off as a direct summand of M . Let N' be any submodule of M such that N makes a direct sum with N' . Take a basis of $N \oplus N'$ and a basis of M such that the first n elements in both are the same and form a basis of N . Then, the matrix of the first basis in terms of the second looks like

$$\begin{pmatrix} I & K \\ 0 & L \end{pmatrix}$$

where I is the $n \times n$ identity matrix. It is clear that the SNF of this matrix will contain I as a submatrix. Thus, we have

Lemma 4.2. *Suppose that M is a free \mathbb{Z} -module of finite rank. Let N be a submodule which splits off as a direct summand of M . Let N' be any submodule of N which makes a direct sum with N . Then, the number of 1s in $\Delta_S(N \oplus N' \hookrightarrow M)$ is at least $\text{rank}(N)$.*

Let's us now say something about the surjectivity of the down maps. Notice that for a differential poset, D_{n+1} is surjective if and only if $\Delta_S(D_{n+1})$ consists of p_n 1s. Now, as the matrix of U_n in the standard basis is transpose to D_{n+1} , $\Delta_S(U_n) = \Delta_S(D_{n+1})$. So, if D_{n+1} is surjective, $\Delta_S(U_n)$ consists of p_n 1s. This, in turn, is equivalent to requiring that U_n has free cokernel. We record this observation as

Lemma 4.3. *For a r -differential poset, the map D_{n+1} is surjective if and only if U_n has free cokernel.*

Remark 4.4. For our main theorem, we assume that the down maps are all surjective. The equivalent condition that the up maps have free cokernel is precisely Conjecture 2.4 in [3]. We also assume a mild modification of Conjecture 2.3 in [3] for all n from some point onwards, and the truth of Conjecture 1.1 for all values of n up to that point. See [3] for motivation of these conjectures.

We are now ready to prove

Theorem 4.5. *Let P be a r -differential poset such that*

- *the down maps are surjective,*
- *there exist some $l \geq 0$ such that $\Delta p_n \geq \Delta p_{n-1-\delta_{r,1}} + 1$ for every $n \geq l + 1$,*
- *the maps DU_n have rational canonical form over \mathbb{Z} for all $n \in \{0, 1, 2, \dots, l\}$.*

Then, the maps DU_n have rational canonical form over \mathbb{Z} for all $n \geq 0$.

Proof. We prove this by induction on n . Base case verification is included in our assumption. For the induction step, we won't assume at the moment that $n \geq l$, and we will indicate when we do.

So, suppose that DU_n has RCF all values up to some $n \geq 0$. We consider \mathbb{Z}^{p_n} and $\mathbb{Z}^{p_{n+1}}$ as $\mathbb{Z}[x]$ -modules, with x -action on \mathbb{Z}^{p_n} induced by DU_n and by UD_{n+1} on $\mathbb{Z}^{p_{n+1}}$. It is then easily seen that U_n and D_{n+1} define $\mathbb{Z}[x]$ -module homomorphisms between \mathbb{Z}^{p_n} and $\mathbb{Z}^{p_{n+1}}$. The induction hypothesis then implies that there exist cyclic $\mathbb{Z}[x]$ -submodules V_1, V_2, \dots, V_m of \mathbb{Z}^{p_n} such that $\mathbb{Z}^{p_n} = \bigoplus_{i=1}^m V_i$ and each $\text{Ann}(V_i)$ is generated by a non-constant monic polynomial $a_i(x)$, which satisfy $a_1(x)|a_2(x)|\dots|a_m(x)$. If v_i generates V_i ,

then $\nu := \cup_{i=1}^m \{x^j v_i\}_{j=0}^{d_i-1}$ is a basis of \mathbb{Z}^{p_n} . By the discussion in §3 of this paper, we also know that $m = \max \{\Delta p_n\}_{j=0}^n$ and

$$a_i(x) = \prod_{\substack{1 \leq j \leq n+1 \\ \Delta p_{n-j+1} \geq m-i+1}} (x - rj).$$

Our goal now is to prove that $\mathbb{Z}^{p_{n+1}}$ has RCF over \mathbb{Z} .

For each v_i , choose a $w_i \in \mathbb{Z}^{p_{n+1}}$ such that $D_{n+1}(w_i) = v_i$. Let W_i be the $\mathbb{Z}[x]$ -submodule generated by w_i . Then, $xW_i = U_{n+1}(V_i) \cong V_i$, which implies that $(xa_i(x)) \subset \text{Ann}(W_i)$. Also, if $p(x)w_i = 0$ for some $p(x) \in \mathbb{Z}[x]$, then $D_{n+1}(p(x)w_i) = p(x)v_i = 0 \implies a_i(x)|p(x) \implies \text{Ann}(W_i) \subset (a_i(x))$. Thus, we have

$$(xa_i(x)) \subset \text{Ann}(W_i) \subset (a_i(x)).$$

Next, let $\omega' := \cup_{i=1}^m \{x^j w_i\}_{j=0}^{d_i-1}$ and $W' = \mathbb{Z}\omega'$. As $D_{n+1}(\omega') = \nu$, and ν is a basis of \mathbb{Z}^{p_n} , we have $\mathbb{Z}^{p_{n+1}} = W' \oplus \ker D_{n+1}$. So, taking κ a basis of $\ker D_{n+1}$, we have $\beta := \omega' \cup \kappa$ a basis for $\mathbb{Z}^{p_{n+1}}$.

$$\begin{array}{ccccccccccc} w_i & \xrightarrow{\quad} & xw_i & \xrightarrow{\quad} & x^2w_i & \xrightarrow{\quad} & x^3w_i & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & x^{d_i}w_i \\ & \searrow D_{n+1} & \uparrow U_n & \searrow D_{n+1} & \uparrow U_n & \searrow D_{n+1} & \uparrow U_n & \searrow D_{n+1} & & \searrow D_{n+1} & \uparrow U_n \\ & & v_i & \xrightarrow{\quad} & xv_i & \xrightarrow{\quad} & x^2v_i & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & x^{d_i-1}v_i \end{array}$$

As DU_n is injective, $\ker D_{n+1} \cap \text{im } U_n = 0$. Let $M = \ker D_{n+1} \oplus \text{im } U_n$. Then, M is the kernel of the surjection $\mathbb{Z}^{p_{n+1}} \mapsto \mathbb{Z}^{p_n} \mapsto \mathbb{Z}^{p_n} / \text{im } DU_n$. Thus,

$$\mathbb{Z}^{p_{n+1}} / M \cong \mathbb{Z}^{p_n} / \text{im } DU_n,$$

and so, the invariant factors of both sides, and hence the non-unit entries of $\Delta_S(M \hookrightarrow \mathbb{Z}^{p_{n+1}})$ and $\Delta_S(DU_n)$ are the same. Since $[DU_n]_\nu$ is in RCF, it is easily seen that $\Delta_S(DU_n) = (1, \dots, 1, a_1(0), a_2(0), \dots, a_m(0))$, which gives

$$\Delta_S(M \hookrightarrow \mathbb{Z}^{p_{n+1}}) = (1, \dots, 1, a_1(0), a_2(0), \dots, a_m(0)),$$

with $p_{n+1} - m$ number of 1s before $a_1(0)$. Since, D_{n+1} is assumed surjective, U_n has free cokernel by Lemma 4.3, which is equivalent to saying that $\text{im } U_n$ splits off as a direct summand of $\mathbb{Z}^{p_{n+1}}$. By Lemma 4.2, we deduce that the number of 1s in $\Delta_S(M \hookrightarrow \mathbb{Z}^{p_{n+1}})$ is at least p_n . We now split off into two cases.

Case 1. $r \geq 2$.

Each of $a_i(0)$ is divisible by r , which means that the number of 1's in $\Delta_S(M \hookrightarrow \mathbb{Z}^{p_{n+1}})$ is exactly $p_{n+1} - m$. This number is greater than p_n , which gives us $\Delta p_{n+1} \geq m \geq \Delta p_{n-i}$ for $0 \leq i \leq n$. As DU_j has RCF over \mathbb{Z} for $j = 0, 1, \dots, n$, and n was only assumed to be greater than or equal to 0, we have $\Delta p_{n+1} \geq \Delta p_n \geq \dots \geq \Delta p_0$. Thus, we may assume that $m = \Delta p_n$.

We now show that $\mathbb{Z}^{p_{n+1}}$ is a direct sum of Δp_{n+1} cyclic $\mathbb{Z}[x]$ -modules, with annihilators $b_i(x) = xa_{i-\epsilon}(x)$ for $1 \leq i \leq \Delta p_{n+1}$, where $\epsilon = \Delta p_{n+1} - \Delta p_n$ and we define $a_i(x) = 1$ for $i \leq 0$.

Let $W := W_1 + W_2 + \dots + W_m$. We wish to prove that this is actually a direct sum. Suppose that $p_1(x)w_1 + p_2(x)w_2 + \dots + p_m(x)w_m = 0$ for some $p_1(x), p_2(x), \dots, p_m(x) \in \mathbb{Z}[x]$. Then, applying the down map D_{n+1} implies $p_1(x)v_1 + p_2(x)v_2 + \dots + p_m(x)v_m = 0 \implies a_i(x)|p_i(x) \implies p_i(x) = c_i a_i(x)$ for some $c_i \in \mathbb{Z}[x]$. However, as $xa_i(x)w_i = 0$, we can assume that $c_i \in \mathbb{Z}$. If all c_i are 0, we are done. If not, then, if c_i have a common factor, we can remove it by dividing by $\gcd(c_1, c_2, \dots, c_m) \neq 0$, so we may further assume that $\gcd(c_1, c_2, \dots, c_m) = 1$. Notice that the coefficient of w_i in the expansion of $c_1 a_1(x)w_1 + \dots + c_m a_m(x)w_m$ is $c_i a_i(0)$, which is divisible by $a_1(0)$, a non-unit as $r|a_1(0)$. So, from this expansion, we find that

$$c_1(a_1(x) - a_1(0))w_1 + \dots + c_m(a_m(x) - a_m(0))w_m = -c_1 a_1(0)w_1 - \dots - c_m a_m(0)w_m$$

is divisible by $a_1(0)$. The left hand side belongs to the \mathbb{Z} -span of xw' and the gcd of the coefficients of this span is still 1. Since $\text{coker } U_n$ is free over \mathbb{Z} and xw' is a basis for $\text{im } U_n$, xw' can be extended to a basis of $\mathbb{Z}^{p_{n+1}}$. Invoking Lemma 4.1, we get a contradiction. Thus, $c_1 = c_2 = \dots = c_m = 0$, and so, we have

$$W = W_1 \oplus W_2 \oplus \dots \oplus W_m.$$

The argument just given shows two more things. First, $\text{Ann}(W_i) = (xa_i(x))$. Second, if we let $\hat{k}_i := a_i(x)w_i$, then, $\hat{k}_i \in \ker x = \ker UD_{n+1} = \ker D_{n+1}$, and $\hat{\kappa} := \{\hat{k}_i\}_{i=1}^m$ is a linearly independent set.

We now assume that $n \geq l$. Notice that since $\Delta p_{n+1} \geq \Delta p_0 = 1$, $\ker D_{n+1}$ is non-trivial. Thus, the choice of w_i was not unique, and all of our work is valid if we replace, say w_1 with $w_1 + k$ for some $k \in \ker D_{n+1}$. Our goal at this stage is that, by tweaking w_i with elements of $\ker D_{n+1}$ if necessary, W splits off as a direct summand of $\mathbb{Z}^{p_{n+1}}$, with the other direct summand a submodule of $\ker D_{n+1}$. We achieve this by showing that for some suitable choice of w_i , $\mathbb{Z}\hat{\kappa}$ splits off as a direct summand of $\ker D_{n+1}$.

We apply the Hermite Canonical Form Theorem for the inclusion $\hat{\kappa} \hookrightarrow \ker D_{n+1}$, but with the basis elements in reverse order. We thus assume that $\kappa = \{k_i\}_{i=1}^{\Delta p_{n+1}}$ is a basis of $\ker D_{n+1}$ such that

$$\begin{aligned} \hat{k}_m &= b_1 k_1 \\ \hat{k}_{m-1} &= b_{1,2} k_1 + b_2 k_2 \\ &\vdots \\ \hat{k}_1 &= b_{1,m} k_1 + b_{2,m} k_2 + \dots + b_m k_m. \end{aligned}$$

Equivalently, the matrix of $\hat{k}_m, \hat{k}_{m-1}, \dots, \hat{k}_1$ is

$$\begin{bmatrix} b_1 & b_{1,2} & b_{1,3} & \dots & b_{1,m} \\ & b_2 & b_{2,3} & \dots & b_{2,m} \\ & & b_3 & \dots & b_{3,m} \\ & & & \ddots & \vdots \\ & & & & b_m \end{bmatrix}$$

where there are $\Delta p_n - m = \Delta p_{n+1} - \Delta p_n \geq 1$ rows of 0s. Notice that adding k to w_i adds $a_i(0)k$ to \hat{k}_i . This gives us the operation of adding $a_i(0)$ times any integer column vector to the $(m-i+1)$ -th column. Row operations correspond to changing the basis of

$\ker D_{n+1}$. Using these two sets of operations, we show that we can make the top $m \times m$ block an identity matrix. This would give us that $\widehat{k}_1, \widehat{k}_2, \dots, \widehat{k}_m$ can be extended to a basis of $\ker D_{n+1}$.

We do this inductively. Suppose i pivots have been turned into one, with $0 \leq i \leq m-1$. So, our matrix looks like

$$\begin{bmatrix} 1 & & b_{1,i+1} & \dots & b_{1,m} \\ & \ddots & \vdots & & \vdots \\ & & 1 & b_{i,i+1} & \dots & b_{i,m} \\ & & & b_{i+1} & \dots & b_{i+1,m} \\ & & & & \ddots & \vdots \\ & & & & & b_m \end{bmatrix}.$$

We now fix the $i+1$ -th pivot. We claim that $\gcd(b_{i+1}, a_{m-i}(0)) = 1$. If p is a prime which divides both b_{i+1} and $a_{m-i}(0)$, it also divides $a_{m-j}(0)$ for $0 \leq j \leq i$ because $a_{m-i}(0)|a_{m-j}(0)$. Thus, p divides

$$\begin{aligned} w &:= b_{i+1}k_{i+1} - a_{m-i}(0)w_{m-i} - \sum_{j=0}^{i-1} b_{j+1,i+1}a_{m-j}(0)w_{m-j} \\ &= \widehat{k}_{m-i} - \sum_{j=1}^i b_{j,i+1}k_j - a_{m-i}(0)w_{m-i} - \sum_{j=0}^{i-1} b_{j,i+1}a_{m-j}(0)w_{m-j} \\ &= \widehat{k}_{m-i} - \sum_{j=0}^{i-1} b_{j+1,i+1}\widehat{k}_{m-j} - a_{m-i}(0)w_{m-i} - \sum_{j=0}^{i-1} b_{j+1,i+1}a_{m-j}(0)w_{m-j} \\ &= (a_{m-i}(x) - a_{m-i}(0))w_{m-i} - \sum_{j=0}^{i-1} b_{j+1,i+1}(a_{m-j}(x) - a_{m-j}(0))w_{m-j} \end{aligned}$$

and this last sum is a linear combination of elements of $x\omega'$, with the coefficient of $x^{d_{m-i}}w_{m-i}$ equal to 1. As $p|w$, w is in span of vectors of $x\omega'$ with gcd of coefficients 1 and $x\omega'$ can be extended to a basis of $\mathbb{Z}^{p_{n+1}}$, by Lemma 4.1, we have a contradiction. Thus, we assume that $\gcd(b_{i+1}, a_{m-i}(0)) = 1$. We now replace w_{m-i} with $w_{m-i} + k_{i+2}$. This replaces \widehat{k}_{m-i} with $\widehat{k}_{m-i} + a_{m-i}(0)k_{i+2}$. By applying row operations to only rows $i+1$ and $i+2$ in a manner that executes the Euclidean Division algorithm for b_{i+1} and $a_{m-i}(0)$, we end up replacing b_{i+1} with 1, which can then be used to remove the entries above it with i row operations. Thus, with the new $\widehat{k}_m, \widehat{k}_{m-1}, \dots, \widehat{k}_1$, the matrix (with some new pivots and $b_{i,j}$'s) is

$$\begin{bmatrix} 1 & & 0 & \dots & b_{1,m} \\ & \ddots & \vdots & & \vdots \\ & & 1 & 0 & \dots & b_{i,m} \\ & & & 1 & \dots & b_{i+1,m} \\ & & & & \ddots & \vdots \\ & & & & & b_m \end{bmatrix}.$$

Thus, one can fix all the pivots. Notice that last element is fixed by using the $m + 1$ -th row, and this is where we make use of the assumption $\Delta p_{n+1} \geq \Delta p_n + 1$.

Now, take κ to be basis in which the first m elements are \widehat{k}_i . Then, as we know that $\beta = \omega' \cup \kappa$ is a basis for $\mathbb{Z}^{p_{n+1}}$, we can replace each \widehat{k}_i with $\widehat{k}_i + (x^{d_i} - a_i(x))w_i = x^{d_i}w_i$. Thus, we obtain a basis of ω of \mathbb{Z}^{p_n} whose elements are from $\cup_{i=1}^m \{w_i, \dots, x^{d_i}w_i\}$ and some elements of $\ker D_{n+1}$. Each $\{w_i, \dots, x^{d_i}w_i\}$ is a basis of W_i , whose annihilators are $(xa_i(x))$. The remaining kernel elements each generate a cyclic module with annihilator (x) . We thus obtain the RCF of $\mathbb{Z}^{p_{n+1}}$, or equivalently, of UD_{n+1} over \mathbb{Z} .

By Theorem 3.1, the matrix of $xI_{p_{n+1}} - [UD_{n+1}]_\omega$ can be brought into SNF by row and column operations. Then, replacing x with $x - r$, we deduce that $xI_{p_{n+1}} - [DU_{n+1}]_\omega$ has SNF over $\mathbb{Z}[x]$ which, again by Theorem 3.1 implies that DU_{n+1} has RCF over \mathbb{Z} . The induction step is complete.

Case 2. $r = 1$.

Let $\eta = \max \{\Delta p_{n-1}, \Delta p_{n-2}, \dots, \Delta p_0\}$. The number of 1's in $\Delta_S(M \hookrightarrow \mathbb{Z}^{p_{n+1}})$ is then $p_{n+1} - \eta$, as η counts the number of polynomials $a_i(x)$ with a non-unit constant term. Since this must be greater than or equal to p_n , we obtain that $\Delta p_{n+1} \geq \eta \geq \Delta p_{n-i}$ for $1 \leq i \leq n$. As DU_j has RCF over \mathbb{Z} for $j = 0, 1, \dots, n$, and n was assumed to be only greater than or equal to 0, we see that $\Delta p_j \geq \Delta p_{j-i-1}$ for $1 \leq i \leq j - 1$ for all j up to $n + 1$. In particular, $m = \max \{\Delta p_n, \Delta p_{n-1}\}$.

If $m = \Delta p_{n-1}$, essentially the same proof as for the previous case gives the RCF of $\mathbb{Z}^{p_{n+1}}$ with a total of Δp_{n+1} submodules, since $a_1(0)$ is still a non-unit. We show what to do when $\delta = \Delta p_n - \Delta p_{n-1} \geq 1$. We have $a_1(x) = a_2(x) = \dots = a_\delta(x) = x - 1$, and $a_i(x)$ have non-unit constant term for $i > \delta$. We also insist that one chooses w_i initially so that $xw_i = w_i$ for $1 \leq i \leq \delta$. Again, the same argument works if we take $W := W_{\delta+1} + \dots + W_m$. That is W is a direct sum of W_i for $i > \delta$, $\text{Ann}(W_i) = (xa_i(x))$ for $i > \delta$, and there is a basis of $\ker D_{n+1}$, whose first $m - \delta$ elements are $a_i(x)w_i$ for $i > \delta$. One can then replace these $m - \delta$ basis elements of $\ker D_{n+1}$ with $x^{d_i}w_i$ for $i > \delta$. Suppose now that the left off submodule K of $\ker D_{n+1}$ which makes a direct summand with W has rank s . Notice that we still have $xw_i = w_i$ for $1 \leq i \leq \delta$, since these weren't tweaked in the process, and that $w_1, w_2, \dots, w_\delta$ extend any basis of $W \oplus K$ to a basis of $\mathbb{Z}^{p_{n+1}}$. Say, k_1, k_2, \dots, k_s is a basis of K . We replace k_1 with $w_\delta + k_1$, k_2 with $w_{\delta-1} + k_2$, until we exhaust one of the lists. The annihilator of the newly tweaked w_i are now $x(x - 1)$. The modules generated by the left off of the untweaked w_i or k_i and the newly tweaked w_i for $i < \delta$, together with the submodules W_i for $i > \delta$ give the RCF of $\mathbb{Z}^{p_{n+1}}$, with a total of $\max \{\Delta p_{n+1}, \Delta p_n\}$ number of submodules. From this point, the induction step is completed in the same manner. \square

Remark 4.6. It seems quite likely that the induction step can be completed without the extra assumption on rank size differences. Notice that the existence RCF of DU_n over \mathbb{Z} implies the bound $\Delta p_{n+1} \geq \Delta p_{n-\delta_{r,1}}$. The difficulty comes when it becomes an equality, since then, there are no zero rows in the matrix of $\widehat{\kappa}$ to fix the last pivot. However, we can still make $m - 1$ pivots equal to 1. The last pivot d_m can be fixed if $d_m \equiv \pm 1 \pmod{a_{i_0}(0)}$, where $a_{i_0}(x)$ is the first annihilator of DU_n which has a non-unit constant. Notice that further tweaking of other w_i would no longer help, since the determinant of this matrix modulo $a_{i_0}(0)$ would be d_m , and unless this is ± 1 , one cannot fix the proof by just tweaking w_i alone.

5. APPLICATIONS

Theorem 5.1. *Let P and Q be differential posets, with rank sizes p_n , q_n respectively. Suppose that $\Delta q_n \geq \Delta q_{n-1}$ for all $n \geq 2$, and that all the down maps of at least one of the posets are surjective. Then, the Conjecture 1.1 holds for $P \times Q$.*

Proof. By Corollary 3.2, it is enough to show that the DU_n maps of $P \times Q$ have RCF over \mathbb{Z} . Notice that $P \times Q$ is a r -differential poset for some $r \geq 2$. It was proved in [3, §4.2] that the up maps of a cartesian product have free cokernel if one of the posets in the product has this property. Casting this in terms of surjectivity of down maps, we conclude that the down maps of $P \times Q$ are surjective.

We denote the rank sizes of $P \times Q$ by pq_n . We know that

$$pq_n = \sum_{i=0}^n q_{n-i} p_i.$$

So,

$$\Delta pq_n - \Delta pq_{n-1} = q_0(\Delta p_n) + \Delta q_1 p_{n-1} + \sum_{i=0}^{n-2} (\Delta q_{n-i} - \Delta q_{n-i-1}) p_i.$$

If Q is a 1-differential poset, then $\Delta q_2 - \Delta q_1 = 1$, so the last summand in the sum contributes a non-zero term p_{n-2} for $n \geq 2$, and if Q is s -differential for $s \geq 2$, $\Delta q_1 = s - 1$ which means that $\Delta q_1 p_{n-1} \geq 1$. All the terms in the sum are non-negative, which means that this sum is always at least 1 for $n \geq 2$. Thus, $\Delta pq_n - \Delta pq_{n-1} \geq 1$ for $n \geq 2$. Additionally, if PQ is r -differential for $r \geq 3$, then $\Delta pq_1 - \Delta pq_0 = r - 2 \geq 1$.

Now, DU_0 is always in RCF. If $r = 2$, The matrix for DU_1 can always be taken to be $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, and one can easily verify that it has RCF over \mathbb{Z} . Taking $l = 1$ in Theorem 4.5, we get the result for $r = 2$. If $r \geq 3$, we can take $l = 0$. \square

Corollary 5.2. *The Conjecture 1.1 is true for \mathbf{Y}^r for every $r \geq 1$.*

Proof. It was proved in [3, §6.1] that the up maps of \mathbf{Y} have free cokernel and hence, the down maps are surjective. We prove that the rank size condition $\Delta p_n \geq \Delta p_{n-\delta_{r,1}-1}$ holds for $n \geq 3$ for $r = 1$ first.

We have $\Delta p_1 = 0$ and $\Delta p_3 = 1$, so the conditions holds for $n = 3$. So, assume $n \geq 4$. We know that $\Delta p_n = p_n - p_{n-1}$ counts the number of partitions of n with no part equal to 1. Let S_n be the set of all such partitions of n . For each partition in S_{n-2} , we can add a 2 to the largest part, and obtain a partition of n in S_n . This injects S_{n-2} in S_n . Thus, $|S_n| \geq |S_{n-2}|$. If n is even, the partition $2, 2, 2, \dots, 2$ with $n/2$ number of 2s cannot be obtained from the injection of S_{n-2} . If n is odd, the partition $3, 2, 2, \dots, 2$ with $\lfloor n/2 \rfloor - 1$ number of 2s is a partition not coming from injection of S_{n-2} . So, we have $|S_n| > |S_{n-2}|$ for $n \geq 4$, and we obtain the desired inequality. One can easily verify that DU_0 , DU_1 and DU_2 have RCF over \mathbb{Z} . Invoking Theorem 4.5 and Corollary 3.2, we get the result.

To prove the result for $r \geq 2$, we argue as follows. In the proof for $r = 1$, we could also have injected S_{n-1} in S_n by adding 1 to the largest part, \mathbf{Y} . This allows us to invoke Theorem 5.1 with $P = \mathbf{Y}^{r-1}$ and $Q = \mathbf{Y}$. \square

Corollary 5.3. *The Conjecture 1.1 holds for $Z(r)$ for all $r \geq 1$.*

Proof. The requirement of surjectivity of down maps was proved in [3, §5]. For rank sizes, we reason as follows. If $r = 1$, notice that $\Delta p_n = f_{n-2}$, where f_n denotes the n -th Fibonacci number. So, $\Delta p_n - \Delta p_{n-2} \geq \Delta p_n - \Delta p_{n-1} = f_{n-2} - f_{n-3} = f_{n-4} \geq 1$ for $n \geq 4$. The base case verification is the same as for \mathbf{Y} .

For $r \geq 2$, we invoke Theorem 5.1 with $P = Z(r - 1)$ and $Q = Z(1)$. \square

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